Word Theory Applied to Musical Scales

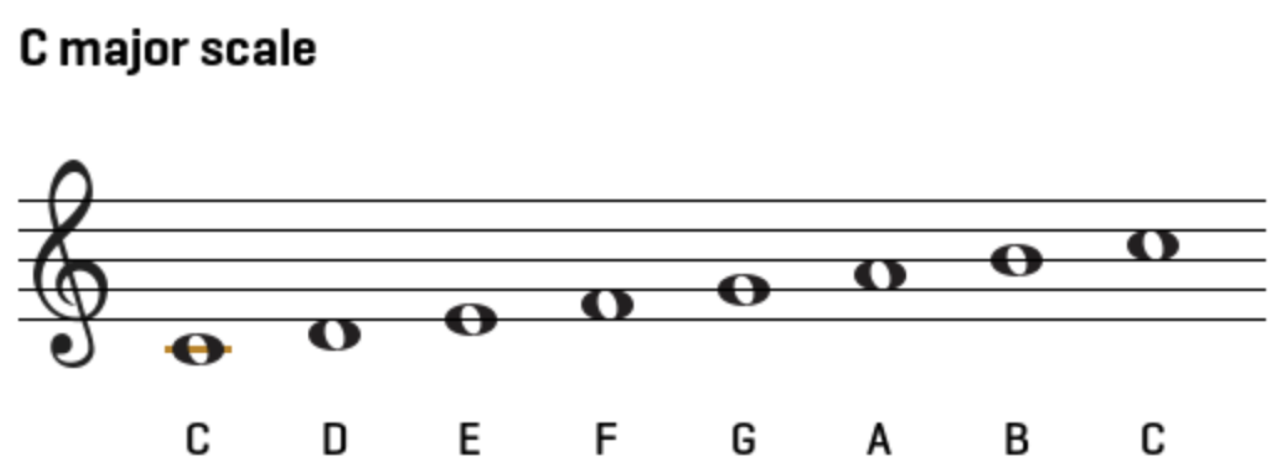
Jackson Rockmael and Tim Guan
University of California Santa Barbara DRP

A Brief History of the Ionian Scale

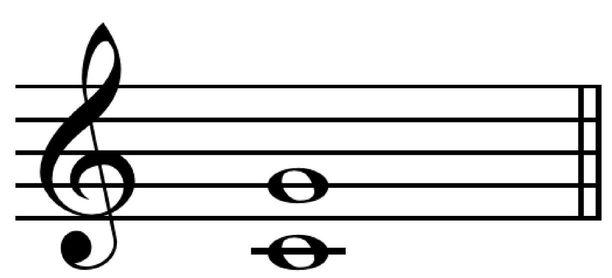
The Ionian scale, known today as the major scale, originated in ancient Greek music and became a cornerstone of Western music due to its bright and harmonious sound. Today, the major scale remains fundamental in classical, pop, rock, and jazz music, underscoring its timeless and universal appeal. Out of all the 792 seven-note scales, we set out to find what makes Ionian unique.

Fundamentals of Music Theory

What is a Scale? A musical scale is a sequence of notes arranged in ascending descending order of pitch within an octave. Scales form the foundation of musical compositions and are used to define the tonality of a piece of music. Each scale is made up of notes that follow a specific pattern of intervals, which are the tonal gaps between the notes.



The Perfect 5th is an interval between 2 notes within the 7-note scale and is considered highly consonant, meaning it sounds stable and pleasant to the ear. This consonance is due to the simple frequency ratio of 3:2 between the two notes, which creates a sense of balance and resolution, making it the most important interval in music. For example, G is the perfect 5th of C.



When looking for the scale that sounds the best, we turn our attention to a scale that has the maximum amount of perfect 5th. This narrows down our search to 7 scales commonly known as, Ionian, Dorian, Phrygian, Lydian, Mixolydian, Aeolian, and Locrian; referred to as the *modes*.

Monoids

The **free monoid** generated by $\{x, y\}$, denoted as $\{x, y\}^*$, is the set of possible finite sequences of x and y and these finite sequences are called strings. $\{x, y\}^*$ is closed under the concatenation of strings and contains the identity, the empty string.
We can represent each musical mode as a sequence of intervals, with each interval either ascending by one or two notes. For example, the Ionian mode can be represented as $'aabaaab'$ where a denotes ascent of two notes and b denotes an ascent of one note. When expressed in this way, each mode is simply a rotation of the others in the sequence.

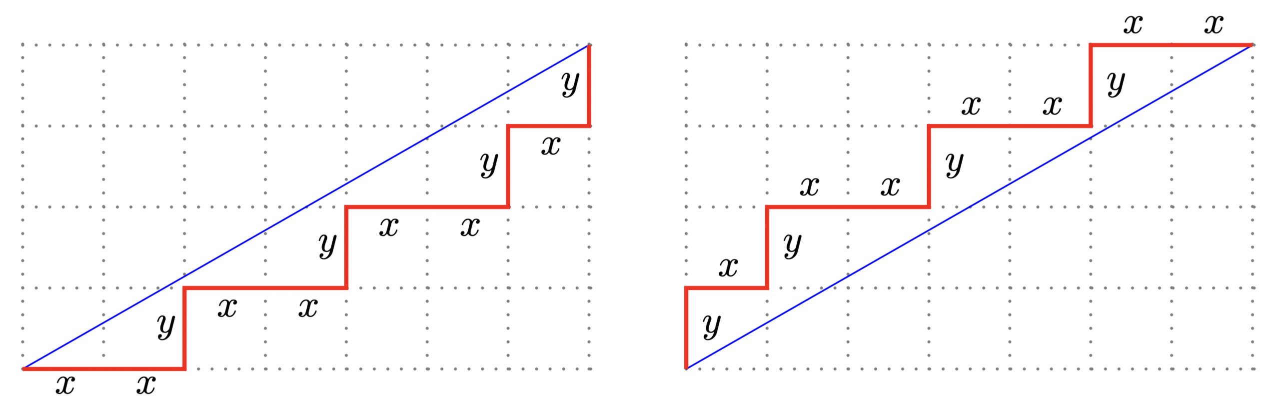
Christoffel Words

The notation $a \perp b$ refers to a and b being relatively prime. Suppose $a, b \in N$ and $a \perp b$. The lower Christoffel path of slope $\frac{b}{a}$ is the path from $(0, 0)$ to (a, b) in the integer lattice $\mathbf{Z} \times \mathbf{Z}$ that satisfies the following two conditions.

(i) The path lies below the line segment that begins at the origin and ends at (a, b).

(ii) The region in the plane enclosed by the path and the line segment contains no other points of $\mathbf{Z} \times \mathbf{Z}$ besides those of the path.

The following diagram represents the upper and lower Christoffel words of slope $\frac{4}{7}$ which are $'yxyxxyxxyxx'$ and $'xxyxxyxxyxy'$ making them strings in $\{x, y\}$.



Definition: Two elements w and w' of $\{a, b\}^*$ are *conjugate* if and only if there exist words u and v such that $w = uv$ and $w' = vu$.

In the illustration below, our words (or scales) are just rotations of each other. This is the case for all conjugates in the free monoid $\{a, b\}^*$. Moreover, in the free group, $\{a, b\}$, the set of all reduced words on the alphabet $\{a, b, a^{-1}, b^{-1}\}$ and the inverse of any word is constructed by taking the reverse spelling and inverting each element. For example, $(aab)^{-1} = b^{-1}a^{-1}a^{-1}$.

Conjugation on Lydian	Result	Mode Name
$b \circ aaabaab \circ b^{-1}$	$baaabaa$	Phrygian
$ab \circ aaabaab \circ b^{-1}a^{-1}$	$abaaaba$	Dorian
$aab \circ aaabaab \circ b^{-1}a^{-1}a^{-1}$	$aabaaab$	Ionian
$baab \circ aabaaab \circ b^{-1}a^{-1}a^{-1}b^{-1}$	$baabaaa$	Locrian
$abaab \circ aabaaab \circ b^{-1}a^{-1}a^{-1}b^{-1}a^{-1}$	$abaabaa$	Aeolian
$aabaab \circ aabaaab \circ b^{-1}a^{-1}a^{-1}a^{-1}b^{-1}a^{-1}$	$aabaaba$	Mixolydian
$aaabaab \circ aabaaab \circ b^{-1}a^{-1}a^{-1}a^{-1}b^{-1}a^{-1}a^{-1}$	$aaabaab$	Lydian

Sturmian Morphisms

Definition: A **Sturmian Morphism** is a monoid homomorphism $\{a, b\}^* \to \{a, b\}^*$ that sends every Christoffel word to a conjugate of a Christoffel word. The set of Sturmian morphisms forms a monoid under function composition. We denote the monoid of Sturmian morphisms by St.

If $A \in St$ and $z_1z_2...z_n \in \{a, b\}^*$, then $A(z_1z_2...z_n) = A(z_1)A(z_2)...A(z_n)$ such that any Sturmian Morphism of $\{a, b\}$ is determined by the images of a and b so we identify A with the ordered pair $(A(a), A(b))$.

Generation of Scales: Using the notation above the monoid, St, of Sturmian morphisms is generated by the following Sturmian Morphisms.

$$G = (a, ab), \tilde{G} = (a, ba), D = (ba, b), \tilde{D} = (ab, b), E = (b, a)$$

Special Sturmian Morphisms

An important sub-monoid of this, St_0, is the monoid generated by **G, G̃, D and G̃**(note the absence of ***E***). St_0 is called the moniod of *special Sturmian Morphisms*, and these play a distinguished role in the *Divider Incidence Theorem*.

Divider Incidence Theorem: In the conjugacy class of a Christoffel word of length n, there are $n - 1$ words that can be obtained as images $f(ab) = f(a)(b) = f(a)f(b)$ of the initial word ab where $f \in St_0$. We separate this word $f(ab)$ into factors giving us a divided word $f(a)|f(b)$.

Mode	Sturmian Representation on (ab)
Ionian	$GGD(ab) = GGD(a)(b) = (aaba)\|(aab)$
Dorian	$G\tilde{G}D(ab) = G\tilde{G}D(a)(b) = (abaa)\|(aba)$
Phrygian	$\tilde{G}\tilde{G}D(ab) = \tilde{G}\tilde{G}D(a)(b) = (baaa)\|(baa)$
Lydian	$GG\tilde{D}(ab) = GG\tilde{D}(a)(b) = (aaab)\|(aab)$
Mixolydian	$G\tilde{G}\tilde{D}(ab) = G\tilde{G}\tilde{D}(a)(b) = (aaba)\|(aba)$
Aeolian	$\tilde{G}\tilde{G}\tilde{D}(ab) = \tilde{G}\tilde{G}\tilde{D}(a)(b) = (abaa)\|(baa)$

Conclusion

The following table gives the six possible diatonic words (or scales) that can be obtained through our *special Sturmian Morphisms.* We should notice that there is one conjugate missing from this list, which is the *Locrian* mode, otherwise known as **'E'**, represented by $'(baab)|(aaa)'$. This is the only conjugate that cannot be generated by $f(ab)$ with $f \in St_0$, which we call the *'bad conjugate.'*

Out of the generators of St_0 we can determine that D and G best preserve the scale integrity. So when choosing a scale out of the 6 modes that are generated by St_0, we would choose the scale generated by D and G, which is Ionian. So, through Christoffel words and Sturmian Morphisms, we can fully characterize Ionian as the universally best 7-note scale.

Acknowledgements

We thank Alexander Sabater for his guidance as well as the UCSB Directed Reading Program and Math Department for the opportunity to work on this project.

References

(1) Berstel, Jean, et al. "Codes and Automata." Université Gustave Eiffel, 2008, www-igm.univ-mlv.fr/ berstel/Articles/2008wordsbookMtlUltimate.pdf.

(2) Fiore, Thomas M. "Music and Mathematics: From Pythagoras to Fractals." University of Michigan-Dearborn, 2006, https://www-personal.umd.umich.edu/ tmfiore/1/musictotal.pdf.

(3) Trotter, Matthias. "Group Theory in Cryptography." University of Chicago, 2009, www.math.uchicago.edu/may/VIGRE/VIGRE2009/REUPapers/Trotter

UC **SANTA BARBARA**

Curvature Done Optimally

Dongming (Merrick) Hua [1]
Mentor: Fabio Ricci [1]

[1]University of California - Santa Barbara
Department of Mathematics - Directed Reading Program 2024

Introduction

Curvature is a central concept in Riemannian geometry, and bounds on the various curvatures of a manifold M translate into useful constraints on the geometry and topology of M. In particular, lower bounds on the Ricci curvature Ric_M of M play a key role in many important theorems.

We discuss an alternate notion of "curvature bounded below by K" for compact Riemannian manifolds, which only involves the distance on the manifold and the volume measure of the manifold. We will show that in the Riemannian setting, a manifold has Ricci curvature $\geq K$ if and only if it satisfies this alternate condition. This new definition does not explicitly use the Riemannian structure, and thus can be generalized to a broader, nonsmooth class of metric measure spaces.

Comparison Geometry

To get an idea of manifolds with $\mathrm{Ric}_M \geq K$, we can look at the model spaces M^n_K of constant sectional curvature K, where

$$M^n_K = \begin{cases} \text{the sphere } S^n(K), & K > 0 \\ \text{Euclidean space } \mathbb{R}^n, & K = 0 \\ \text{hyperbolic space } \mathbb{H}^n(K), & K < 0. \end{cases}$$

Intuitively, in positive curvature geodesics diverge then converge, in zero curvature they diverge at a constant rate, and in negative curvature they diverge increasingly rapidly.

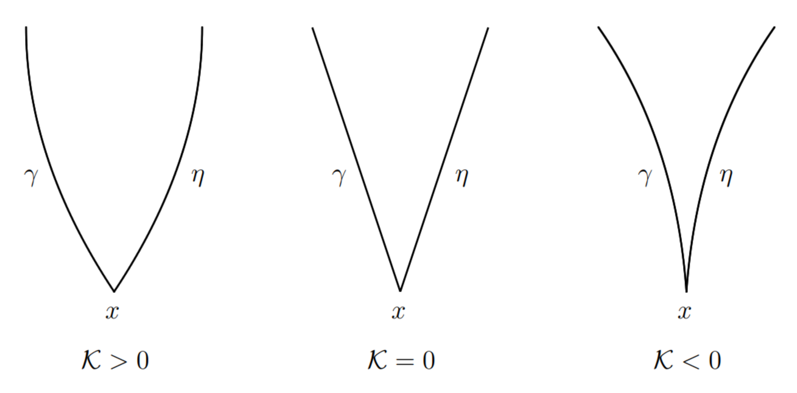
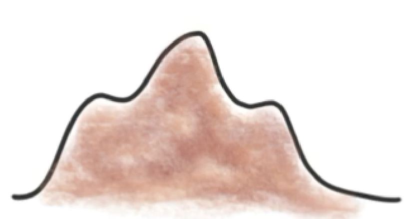
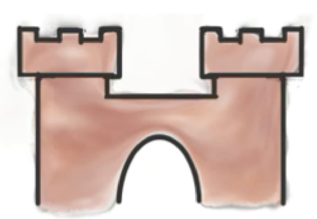


Figure 1. Geodesics in positive, zero, and negative curvature. Image from [3].

Optimal Transport

Optimal transport studies the most efficient way to transport some amount of mass from one configuration to another, such as moving a pile of sand to build a sandcastle:

We can view the configurations of masses as probability measures μ and ν on M and measure efficiency via minimizing the cost

$$\int_M d(x, T(x))^2 \mathrm{d}\mu(x)$$

over maps $T\colon M \to M$ satisfying $T_\sharp \mu = \nu$. McCann showed that if M is compact and $\mu = \rho_0 \,\mathrm{vol}, \nu = \rho_1 \,\mathrm{vol}$, where vol is the normalized volume measure on M, then there exists a unique **optimal transport map** T minimizing the above cost. Moreover, T is of the form

$$T(x) = \exp_x(\nabla\varphi(x))$$

for some semiconvex $\varphi\colon M \to \mathbb{R}$, and the Jacobian determinant of T at x is equal to $\rho_0(x)/\rho_1(T(x))$ μ-almost everywhere.

The Wasserstein 2-Distance

Let $\mathcal{P}_2^{ac}(M)$ be the set of all probability measures on a compact manifold M which are **absolutely continuous** with respect to vol, meaning measures for which we can write $\mu = \rho \,\mathrm{vol}$ for some density ρ. We can give this space a metric by defining

$$W_2(\mu, \nu) = \left(\int_M d(x, T(x))^2 \mathrm{d}\mu(x) \right)^{\frac{1}{2}},$$

where T is the optimal transport map from μ to ν. This distance is called the **2-Wasserstein distance**, and can be defined more generally for metric measure spaces via an alternate formulation of the optimal transport problem.

By the work of McCann, for any two measures $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}$ there is a unique Wasserstein geodesic $(\mu_t)_{0\leq t\leq 1}$ between them, so that $W_2(\mu_s, \mu_t) = |t-s| W_2(\mu_0, \mu_1)$ for all $0 \leq s, t \leq 1$. Moreover, there must exist $T\colon [0,1] \times M \to M$ given by

$$T(t, x) = \exp_x(t\nabla\varphi(x)),$$

so that for each $t \in [0,1]$, the map $T_t\colon M \to M$ defined by $T_t(x) = T(t,x)$ is the optimal transport map from μ_0 to μ_t. Therefore, we can write $\mu_t = (T_t)_\sharp \mu_0$, and for each t the Jacobian determinant of T_t at x is $\rho_0(x)/\rho_t(T_t(x))$ μ_0-almost everywhere. Observe that by properties of the exponential map, $d(x, T_1(x)) = |\nabla\varphi(x)|$ for all x, hence

$$W_2(\mu_0, \mu_1)^2 = \int_M |\nabla\varphi|^2 \mathrm{d}\mu.$$

Optimal Transport Maps and Ricci Curvature

The key connection between optimal transport on M and the Ricci curvature of M is that Ricci curvature features in a differential inequality for the Jacobian determinant of the map $T(t,x) = \exp_x(t\nabla\varphi(x))$.

Let $J_t(x)$ be the Jacobian of T_t at x, and let $\mathcal{J}_t(x) = \det J_t(x)$. We can write $\mathcal{J}_t(x)$ in terms of Jacobi fields along the geodesic $\gamma(t) = \exp_x(t\nabla\varphi(x))$. Then, via the Jacobi equation and the Cauchy-Schwarz inequality, we obtain the inequality

$$\frac{\mathcal{J}_t''}{\mathcal{J}_t} - \left(\frac{\mathcal{J}_t'}{\mathcal{J}_t}\right)^2 \leq \frac{\mathcal{J}_t''}{\mathcal{J}_t} - \left(1 - \frac{1}{n}\right)\left(\frac{\mathcal{J}_t'}{\mathcal{J}_t}\right)^2 \leq -\mathrm{Ric}(\nabla\varphi, \nabla\varphi).$$

Curvature Bounded Below by K

For $\mu \in \mathcal{P}_2^{ac}(M)$, define the **entropy** of μ by

$$H(\mu) = \int_M \rho \log \rho \,\mathrm{d}\,\mathrm{vol},$$

where $\mu = \rho \,\mathrm{vol}$. We say M has **curvature bounded below by K** if for any measures $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(M)$, the unique Wasserstein geodesic $(\mu_t)_{0\leq t\leq 1}$ satisfies

$$H(\mu_t) \leq (1-t)H(\mu_0) + tH(\mu_1) - K\frac{t(1-t)}{2} W_2(\mu_0, \mu_1)^2$$

for all $0 \leq t \leq 1$.

References

[1] Luigi Ambrosio and Nicola Gigli. *A User's Guide to Optimal Transport*. Springer, Berlin, Heidelberg, 2013.

[2] Xianzhe Dai and Guofang Wei. *Comparison geometry for ricci curvature*.

[3] Shin-ichi Ohta. Ricci curvature, entropy and optimal transport. 2014.

[4] Karl-Theodor Sturm. On the geometry of metric measure spaces. 2006.

[5] Cedric Villani. *Optimal Transport: Old and New*. Springer, Berlin, Heidelberg, 2008.

The Main Theorem

Theorem 1 (Equivalence of $\mathrm{Ric}_M \geq K$ and Curvature Bounded Below By K). *A compact Riemannian manifold M^n satisfies $\mathrm{Ric}_M \geq K$ if and only if it has curvature bounded below by K.*

Proof. First, suppose $\mathrm{Ric}_M \geq K$. Let $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(M)$ be arbitrary, and let $T(t,x) = \exp_x(t\nabla\varphi(x))$ the map associated with the unique Wasserstein geodesic $(\mu_t)_{0\leq t\leq 1}$. Then we have $\rho_t(T_t(x)) = \rho_0(x)/\mathcal{J}_t(x)$, and so by this change of variables we have

$$\begin{aligned} \frac{d}{dt}H(\mu_t) &= \frac{d}{dt}\int_M \frac{\rho_0}{\mathcal{J}_t} \log\frac{\rho_0}{\mathcal{J}_t} \mathcal{J}_t \mathrm{d}\,\mathrm{vol} = -\int_M \frac{\mathcal{J}_t'}{\mathcal{J}_t}\rho_0 \mathrm{d}\,\mathrm{vol}, \\ \frac{d^2}{dt^2}H(\mu_t) &= -\int_M \left(\frac{\mathcal{J}_t''}{\mathcal{J}_t} - \left(\frac{\mathcal{J}_t'}{\mathcal{J}_t}\right)^2\right)\rho_0 \mathrm{d}\,\mathrm{vol}\,. \end{aligned}$$

By the differential inequality for $\mathcal{J}$, we have

$$\frac{d^2}{dt^2}H(\mu_t) \geq \int_M \mathrm{Ric}(\nabla\varphi, \nabla\varphi)\mathrm{d}\,\mathrm{vol} \geq K\int_M |\nabla\varphi|^2 \mathrm{d}\,\mathrm{vol} = KW_2(\mu_0, \mu_1)^2.$$

Defining $f(t) = H(\mu_t) + K\frac{t(1-t)}{2}W_2(\mu_0,\mu_1)^2$, we see that $\frac{d^2}{dt^2}f(t) = \frac{d^2}{dt^2}H(\mu_t) - KW_2(\mu_0,\mu_1)^2 \geq 0$. Therefore f is convex, and for any $t \in [0,1]$ we have

$$H(\mu_t) + K\frac{t(1-t)}{2}W_2(\mu_0,\mu_1)^2 = f(t) \leq (1-t)f(0) + tf(1) = (1-t)H(\mu_0) + tH(\mu_1),$$

as desired.

The other direction is more complicated, so we only sketch an outline here. A more detailed account can be found in [5]. Suppose M has curvature bounded below by K. Let $x \in M$, $v \in T_xM$ be arbitrary. Take μ_0 to be the normalized volume measure on a small ball $B_\varepsilon(x)$, and take the transport map to be $T_t(x) = \exp_x(t\delta\nabla\varphi(x))$ for some small δ and suitable φ satisfying $\nabla(\varphi(x_0)) = v$. From this we obtain a Wasserstein geodesic $(\mu_t)_{0\leq t\leq 1}$ given by $\mu_t = (T_t)_\sharp\mu_0$, and choosing ε, δ, and φ carefully, the inequality for $H(\mu_t)$ gives us the desired Ricci curvature bound $\mathrm{Ric}(v,v) \geq K|v|^2$. □

Brunn-Minkowski

One geometric consequence of curvature $\geq K$ is the following Brunn-Minkowski inequality, generalizing the classic Brunn-Minkowski inequality in $\mathbb{R}^n$.

Theorem 2 (Generalized Brunn-Minkowski). *Suppose that M has curvature bounded below by K. For nonempty, compact $A_0, A_1 \subseteq M$ and $t \in (0,1)$, let A_t be the set of all points $\gamma(t)$, where γ runs over all unit-length geodesics with $\gamma(0) \in A_0, \gamma(1) \in A_1$. Then*

$$\ln \mathrm{Vol}(A_t) \geq (1-t)\ln \mathrm{Vol}(A_0) + t\ln \mathrm{Vol}(A_1) + K\frac{t(1-t)}{2}d(A_0, A_1)^2.$$

The following diagram depicts this inequality in the case $K > 0$, reflecting the fact that geodesics spread out and then return back together in positive curvature.

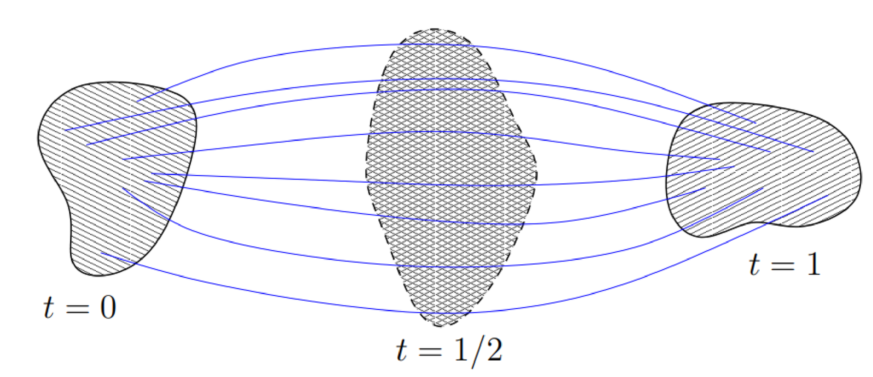


Figure 2. Brunn-Minkowski when $K > 0$. Image from [5].

This inequality can be improved if we introduce a $CD(K,N)$ condition, which encapsulates both "curvature bounded below by K" and "dimension bounded above by N", and in fact the validity of this improved inequality is equivalent to the $CD(K,N)$ condition.

Kirby's Dream 3-Manifold

Miranda Jiang [1] Mentored by Sanjay Kumar [1]

[1]University of California - Santa Barbara
Department of Mathematics - Directed Reading Program 2024

Objective

Welcome to Kirby's dream land. Your goal is to distinguish 3-manifolds through the Kirby Calculus!

Background

The sphere S^3 can be obtained by gluing together two solid tori along the homeomorphism on their boundaries that interchanges longitudes and meridians.

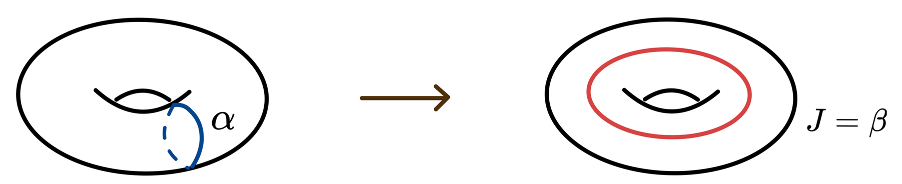


Figure 1. Demonstration of S^3

In fact, any orientable 3-manifold M^3 may be obtained by cutting out some solid tori from the 3-sphere S^3 and then pasting them back in, but along different homeomorphisms of their boundaries. This process is called a **surgery** on S^3.

We can describe the resulting 3-manifold entirely by the image of the meridian α under the attaching homeomorphism of the boundary torus $S^1 \times S^1$. Suppose that the meridian is sent to the curve $J = p\alpha + q\beta$, then J is the closed curve that winds around the boundary torus p times around the meridian and q times around the longitude.

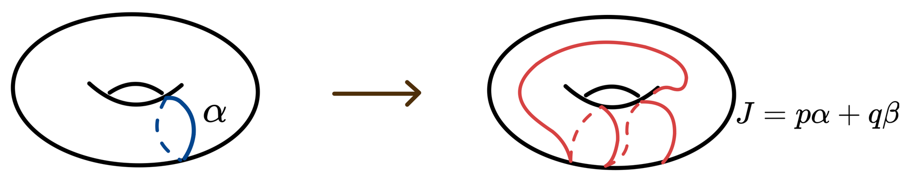


Figure 2. The meridian α is being sent to the curve J

The **framing** of the trivial knot is a rational number $r = p/q$ that determines the surgery of the 3-sphere.

We will consider the case of **integer surgery**, in which we choose q to be 1 so that the framing r is an integer. In fact, any knot diagrams has a natural framing coming from the number of times the normal vector turns when we draw the knot, called the **blackboard framing**. You can think of it as adding twists or kinks to the knot.

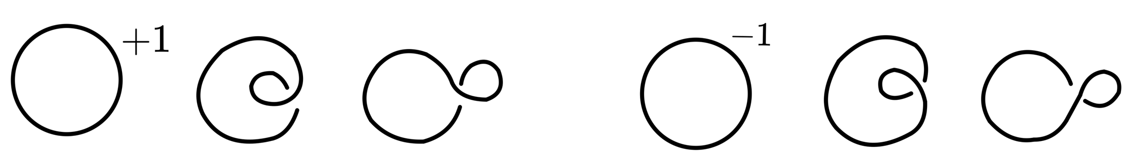


Figure 3. trivial knot with framing +1 and -1, denoted U_+ and U_-, respectively

> ***Theorem*** (Dehn-Lickorish): *Any compact orientable 3-manifold without boundary can be obtained from the sphere S^3 by integer surgery on a framed link.*

Acknowledgements

Huge thanks to my mentor, Sanjay Kumar, for his continuous support during times of ups (the Eureka moments!) and downs (the painful struggles). His enthusiasm deeply inspired me. Thanks to the DRP committee for making this possible!

Thanks to Robion Kirby for all of his good math especially the Kirby Calculus and thank you Nintendo for developing the epic game "Kirby's Dream Land" which made this little poster possible.

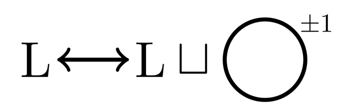
Happy problem-solving!

The Kirby Calculus

Manifolds of the same type can have extremely varied presentations by framed links. We want to find a systematic way to modify framed links so that they represent the same manifold. Here are your allowed moves:

1. The Kirby Moves:

- **The First Kirby Move**: Adding (or deleting) an unknotted circle with framing +- 1 that is unlinked with the other component of the given framed link $L \in S^3$.



- **The Second Kirby Move *("handle-slide")***: In a given framed link diagram with two distinguished unlinked components C and K, with framing indices n and k, respectively, we can slide the first curve C over K so that it encircles K and picking up the framing of K, while leaving the other components unchanged.

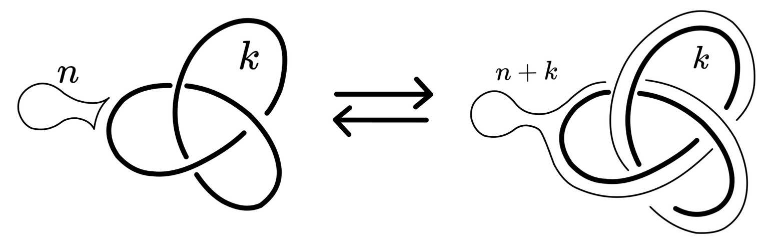


Figure 4. Kirby II

2. The Reidemeister Moves:

- **The Modified First Reidemeister Move:** The original first reidemeister move for knot diagrams involves resolving kinks. In the context of framed links, resolving kinks will result in a change of framing by ± 1. But we can, on the other hand, resolving two kinks with opposite orientations.
- **The Second and Third Reidemeister Move:**

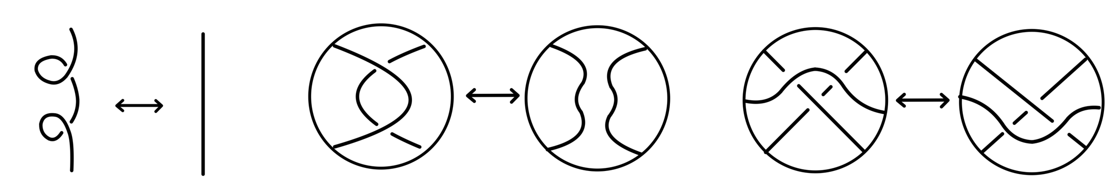


Figure 5. The double twist move Ω_1', Ω_2, and Ω_3

3. **Planer Isotopies:** The action of "smoothing out" the curves.

> ***Theorem*** *(The Kirby Calculus): Two framed links in S^3 produce the same 3-manifold if and only if they can be obtained from each other by a finite sequence of the Kirby moves, the moves Ω_1', Ω_2, and Ω_3 and planer isotopies.*

Example

The following framed link diagrams represent the same 3-manifolds. They can be obtained from each other by a finite sequence of the Kirby moves and the Reidemeister moves.

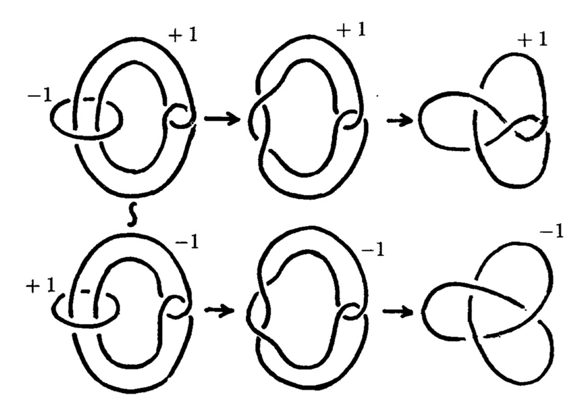


Figure 6. Two modifications of the Whitehead link

Applications

Analogous to building knot invariants with the Jones polynomial, we can build 3-manifold invariants with the Kirby calculus. An example would be the complex-valued 3-manifold invariant $I(D)$ that arises from the **Temperley-Lieb algebra** TL_n. The idea is as follows: given a framed link presentation of a 3-manifold, we can compute a polynomial by means similar to the Jones polynomial, and with a certain choice of coefficients (primitive $4rth$ root of unity), we obtain a complex number that stays invariant under both the Kirby moves and the Reidemeister moves.

We first define the Jones-Wenzl idempotent $f^{(n)}$ by a recursive formula:

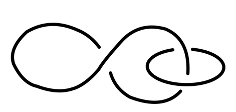
$$f^{(n+1)} = f_1^{(n)} - \frac{\Delta_{(n-1)}}{\Delta_n}(f_1^n e_n f_1^{(n)})$$

where f_1^n is $f^{(n)}$ with a strand added on top, and e_n is the generator of TL_n

We then define the element $\omega = \sum_{n=0}^{r-2} \Delta_n S_n(\alpha)$, where $S_n(\alpha)$ is the image of the closure of $f^{(n)}$ under the map $TL_n \to S(S^1 \times I)$, and Δ_n is a complex number obtained from computing the polynomial of the closure of $f^{(n)}$ under the map $TL_n \to S(\mathbb{R}^2)$

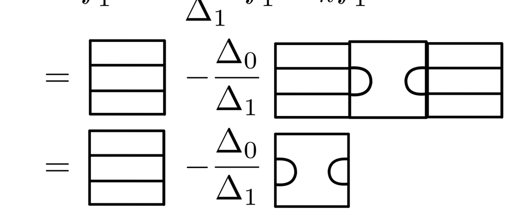
> ***Theorem***: *Suppose that a closed oriented 3-manifold M is obtained by surgery on a framed link represented by a planar diagram D. Let b_+ be the number of positive eigenvalues and b_- be the number of negative eigenvalues of the linking matrix of this link. Suppose $r > 3$ and that A is a primitive $4rth$ root of unity. Then*
>
> $$I(D) = \langle \omega, \dots \omega \rangle_D \langle \omega \rangle_{U_+}^{-b_+} \langle \omega \rangle_{U_-}^{-b_-}$$
>
> *is a well-defined invariant of M.*

Let's do an example.

Example Let $r = 4$, and let D to be this diagram: 

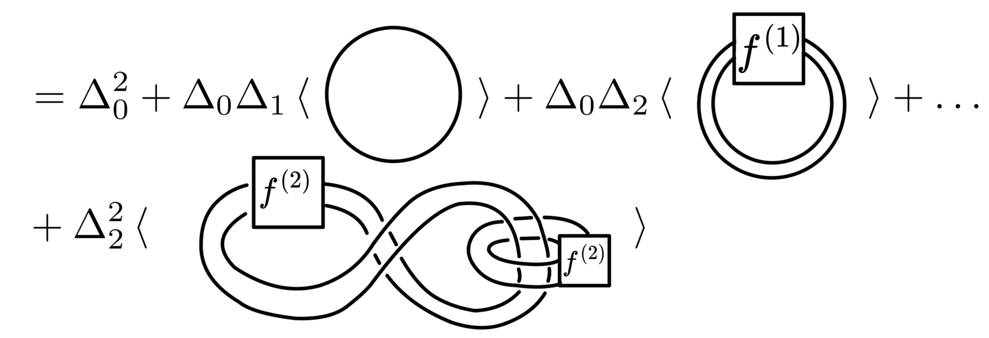
We will compute $f^{(2)}$ first:

$$f^{(2)} = f_1^{(1)} - \frac{\Delta_0}{\Delta_1} f_1^{(1)} e_n f_1^{(1)}$$



Then $\omega = \sum_{n=0}^{2} \Delta_n S_n(\alpha) = \Delta_0\alpha_0 + \Delta_1\alpha_1 + \Delta_2\alpha_2$

$$\begin{aligned}\langle \omega, \omega \rangle_D &= \langle \Delta_0\alpha_0 + \Delta_1\alpha_1 + \Delta_2\alpha_2, \Delta_0\alpha_0 + \Delta_1\alpha_1 + \Delta_2\alpha_2 \rangle_D \\ &= \Delta_0^2 + \Delta_0\Delta_1 \langle \alpha_0, \alpha_1 \rangle_D + \Delta_0\Delta_2 \langle \alpha_0, \alpha_2 \rangle_D + \dots + \Delta_2^2 \langle \alpha_2, \alpha_2 \rangle_D\end{aligned}$$



Plug in $f^{(n)}$ into the square, and do the same operation to $\langle \omega \rangle_{U_+}^{-b_+} \langle \omega \rangle_{U_-}^{-b_-}$, we obtain the desired 3-manifold invariant $I(D)$.

References

[1] A. B. Sossinsky V. V. Prasolov. *Knots, Links, Braids and 3-Manifolds: An Introduction to the New Invariants in Low-Dimensional Topology*. American Mathematical Society, 1997.

[2] W. B. Raymond Lickorish. *An Introduction to Knot Theory*. Springer New York, NY, 1997.